NOTE ON THE ADJUGATE OF BEZOUT'S ELIMINANT OF TWO BINARY QUANTICS.

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(1) It is manifest that the two equations

$$\left.\begin{array}{l} a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0 \\ b_0x^4 + b_1x^3 + b_2x^2 + b_3x + b_4 = 0 \end{array}\right\}$$

give immediate rise to the four

$$\begin{vmatrix} a_0 & a_1x^3 + a_2x^2 + a_3x + a_4 \\ b_0 & b_1x^3 + b_2x^2 + b_3x + b_4 \end{vmatrix} = 0,$$

$$\begin{vmatrix} a_0x + a_1 & a_2x^2 + a_3x + a_4 \\ b_0x + b_1 & b_2x^2 + b_3x + b_4 \end{vmatrix} = 0,$$

$$\begin{vmatrix} a_0x^2 + a_1x + a_2 & a_3x + a_4 \\ b_0x^2 + b_1x + b_2 & b_3x + b_4 \end{vmatrix} = 0,$$

$$\begin{vmatrix} a_0x^3 + a_1x^2 + a_2x + a_3 & a_4 \\ b_0x^3 + b_1x^2 + b_2x + b_3 & b_4 \end{vmatrix} = 0;$$

and that these when expressed explicitly as cubics in x furnish us at once with

$$\begin{vmatrix} |a_0b_1| & |a_0b_2| & |a_0b_3| & |a_0b_4| \\ |a_0b_2| & |a_0b_3| + |a_1b_2| & |a_0b_4| + |a_1b_3| & |a_1b_4| \\ |a_0b_3| & |a_0b_4| + |a_1b_3| & |a_1b_4| + |a_2b_3 & |a_2b_4| \\ |a_0b_4| & |a_1b_4| & |a_2b_4| & |a_3b_4| \end{vmatrix} = 0,$$

the left-hand member of which is known as Bezout's eliminant.

(2) To the adjugate of this eliminant considerable study has been given since Jacobi first drew attention to it ('Crelle's Journ.," xv, pp. 101–124). Unfortunately there has been no simple mode of expressing its elements, which rapidly increase in complexity with the degree of the determinant. Taking advantage of the fact much later established that the primary minors of Bezout's eliminant have equivalents among the secondary minors of Sylvester's eliminant, I have succeeded in obtaining for the adjugate an expression whose law of formation is perfectly simple. Before stating it, it is necessary to recall the fact that the most convenient form of Sylvester's eliminant is that in which the rows of b's follow the opposite order of the rows of a's, being in the case which we are considering

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & \cdot & \cdot & \cdot \\ \cdot & a_0 & a_1 & a_2 & a_3 & a_4 & \cdot & \cdot \\ \cdot & \cdot & a_0 & a_1 & a_2 & a_3 & a_4 & \cdot \\ \cdot & \cdot & \cdot & a_0 & a_1 & a_2 & a_3 & a_4 \\ \cdot & \cdot & \cdot & b_0 & b_1 & b_2 & b_3 & b_4 \\ \cdot & \cdot & b_0 & b_1 & b_2 & b_3 & b_4 & \cdot \\ \cdot & b_0 & b_1 & b_2 & b_3 & b_4 & \cdot & \cdot \\ b_0 & b_1 & b_2 & b_3 & b_4 & \cdot & \cdot & \cdot \end{vmatrix};$$

and that the most convenient way of specifying any minor of this eliminant is by giving the numbers of the rows and columns from which the minor is taken; for example,

$$\begin{vmatrix} 237 \\ 357 \end{vmatrix} \text{ standing for } \begin{vmatrix} a_1 & a_3 & \cdot \\ a_0 & a_2 & a_4 \\ b_1 & b_3 & \cdot \end{vmatrix}.$$

(3) *The adjugate of Bezout's eliminant of*

$$a_0x^4 + a_1x^3 + \ldots + a_4 = 0, \quad b_0x^4 + b_1x^3 + \ldots + b_4 = 0$$

is

$$\begin{vmatrix} \mu_1 & \mu_2 & \mu_3 & \mu_4 \\ \mu_2 & \mu_3 & \mu_4 & \mu_5 \\ \mu_3 & \mu_4 & \mu_5 & \mu_6 \\ \mu_4 & \mu_5 & \mu_6 & \mu_7 \end{vmatrix}$$

where

$$\mu_1 = \begin{vmatrix} 234567 \\ 345678 \end{vmatrix}, \quad \mu_2 = \begin{vmatrix} 234567 \\ 245678 \end{vmatrix}, \quad \ldots\ldots$$

the row-numbers of μ_r *being* 234567 *and the column-numbers being obtained by deleting the digit* $r + 1$ *from* 2345678.

To prove the identity of any one of the μ's with the appropriate primary minor, we may follow the procedure laid down by Le Paige in 1880 ('Comptes Rendus' .... Paris, xc, pp. 1210–1212), namely, multiply the μ in question by such another form of itself as will give the square of the desired equivalent; for example,

$$-\mu_1^2 = -\begin{vmatrix} 234567 \\ 345678 \end{vmatrix}^2 = -\begin{vmatrix} 234567 \\ 345678 \end{vmatrix} \cdot \begin{vmatrix} 567234 \\ 678345 \end{vmatrix},$$

$$= \begin{vmatrix} 234\bar{5}\bar{6}\bar{7} \\ 345678 \end{vmatrix} \cdot \begin{vmatrix} 567234 \\ 678345 \end{vmatrix},$$

where the minus above a row-number indicates change of sign in the row; and columnwise multiplication now gives

$$-\begin{vmatrix} |a_0b_3| + |a_1b_2| & |a_0b_4| + |a_1b_3| & |a_1b_4| \\ |a_0b_4| + |a_1b_3| & |a_1b_4| + |a_2b_3| & |a_2b_4| \\ |a_1b_4| & |a_2b_4| & |a_3b_4| \end{vmatrix}^2,$$

as desired.

It will be found, however, less stale and more effective to multiply the determinant

$$\begin{vmatrix} 1 & \cdot & \cdot & \cdot & \cdot & -b_0 \\ \cdot & 1 & \cdot & \cdot & -b_0 & -b_1 \\ \cdot & \cdot & 1 & -b_0 & -b_1 & -b_2 \\ \cdot & \cdot & \cdot & a_0 & a_1 & a_2 \\ \cdot & \cdot & \cdot & \cdot & a_0 & a_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & a_0 \end{vmatrix}$$

columnwise by the 6-by-7 array $\begin{Vmatrix} 234567 \\ 2345678 \end{Vmatrix}$, for then we obtain

$$a_0{}^3 \cdot \begin{Vmatrix} 234567 \\ 2345678 \end{Vmatrix}$$

$$= \begin{Vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & \cdot & \cdot \\ \cdot & a_0 & a_1 & a_2 & a_3 & a_4 & \cdot \\ \cdot & \cdot & a_0 & a_1 & a_2 & a_3 & a_4 \\ \cdot & \cdot & \cdot & |a_0b_1| & |a_0b_2| & |a_0b_3| & |a_0b_4| \\ \cdot & \cdot & \cdot & |a_0b_2| & |a_0b_3| + |a_1b_2 & |a_0b_4| + a_1b_3| & |a_1b_4| \\ \cdot & \cdot & \cdot & |a_0b_3| & |a_0b_4| + |a_1b_3| & |a_1b_4| + |a_2b_3| & |a_2b_4| \end{Vmatrix},$$

the seven results of which give equivalents for all the μ's, and in four cases (μ_4, μ_5, μ_6, μ_7) the exact equivalents wanted.

(4) What has just been established for the eliminant of the fourth order holds equally for any other order, and the mode of proof is quite general. In case of any doubt about the formation of the μ's, it need only be added that when the eliminant is of the n^{th} order, the row-numbers of μ_r are $2, 3, 4, \ldots, 2n-1$, and the column-numbers are obtained from $2, 3, 4, \ldots, 2n-1, 2n$.

(5) Le Paige's process, mentioned in § 3, led subsequent writers to the discovery of the fact that Bezout's eliminant is variously expressible as the result of multiplying n columns of Sylvester's eliminant by a transformation of the other n columns.* This suggests the making of a similar

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* So far as I am at present aware, the first publication of this is due to H. W. Tyler ('Sitzungst. d. phys.-med Soc. zu Erlangen,' xxiii, pp. 33–128). He makes an oversight, however, in saying that the first set of n columns must be consecutive, and therefore in saying that the number of different pairs of factors is $n+1$. The number in question is either n or $2n$, according to the point of view. Thus, in the case where n is 3 there are the following equivalents of Bezout's eliminant:

$$\begin{array}{ll} \begin{Vmatrix} 123456 \\ 123 \end{Vmatrix} \cdot \begin{Vmatrix} 654\overline{321} \\ 456 \end{Vmatrix}, & \begin{Vmatrix} 123456 \\ 456 \end{Vmatrix} \cdot \begin{Vmatrix} \overline{654}321 \\ 123 \end{Vmatrix}, \\ \begin{Vmatrix} 123456 \\ 234 \end{Vmatrix} \cdot \begin{Vmatrix} \overline{4}65132 \\ 156 \end{Vmatrix}, & \begin{Vmatrix} 123456 \\ 156 \end{Vmatrix} \cdot \begin{Vmatrix} 4\overline{651}32 \\ 234 \end{Vmatrix}, \\ \begin{Vmatrix} 123456 \\ 345 \end{Vmatrix} \cdot \begin{Vmatrix} \overline{54}621\overline{3} \\ 126 \end{Vmatrix}, & \begin{Vmatrix} 123456 \\ 126 \end{Vmatrix} \cdot \begin{Vmatrix} 54\overline{621}3 \\ 345 \end{Vmatrix}, \end{array}$$

the last two in the second column being evidently as worthy of enumeration as the first in the column.

deduction in regard to the primary minors of the eliminant, our first proof regarding μ, in § 3, giving us at once

$$\left\Vert\begin{matrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ \cdot & a_0 & a_1 \\ \cdot & -b_0 & -b_1 \\ -b_0 & -b_1 & -b_2 \\ -b_1 & -b_2 & -b_3 \end{matrix}\right\Vert \cdot \left\Vert\begin{matrix} b_2 & b_3 & b_4 \\ b_3 & b_4 & \cdot \\ b_4 & \cdot & \cdot \\ a_4 & \cdot & \cdot \\ a_3 & a_4 & \cdot \\ a_2 & a_3 & a_4 \end{matrix}\right\Vert = \begin{vmatrix} |a_0b_3| + |a_1b_2| & |a_0b_4| + |a_1b_3| & |a_1b_4| \\ |a_0b_4| + |a_1b_3| & |a_1b_4| + |a_2b_3| & |a_2b_4| \\ |a_1b_4| & |a_2b_4| & |a_3b_4| \end{vmatrix},$$

i. e.

$$\left\Vert\begin{array}{l} 2345\overline{67} \\ 345 \end{array}\right\Vert \cdot \left\Vert\begin{array}{l} 567234 \\ 678 \end{array}\right\Vert = \mu_1;$$

and at the same time

$$\left\Vert\begin{array}{r} 2345\overline{67} \\ 678 \end{array}\right\Vert \cdot \left\Vert\begin{array}{r} 567234 \\ 345 \end{array}\right\Vert = -\mu_1.$$

The theorem obtained as the result of this suggestion is that *the adjugate of Bezout's eliminant of*

$$a_0x^4 + a_1x^3 + \ldots + a_4 = 0, \quad b_0x^4 + b_1x^3 + \ldots + b_4 = 0$$

is

$$\begin{vmatrix} \nu_1 & \nu_2 & \nu_3 & \nu_4 \\ \nu_2 & \nu_3 & \nu_4 & \nu_5 \\ \nu_3 & \nu_4 & \nu_5 & \nu_6 \\ \nu_4 & \nu_5 & \nu_6 & \nu_7 \end{vmatrix},$$

where

$$\nu_1 = \left\Vert\begin{array}{l} 234\overline{567} \\ 345 \end{array}\right\Vert \cdot \left\Vert\begin{array}{l} 567234 \\ 678 \end{array}\right\Vert,$$

$$\nu_2 = \left\Vert\begin{array}{l} 2345\overline{67} \\ 245 \end{array}\right\Vert \cdot \left\Vert\begin{array}{l} 567234 \\ 678 \end{array}\right\Vert,$$

. . . .

and the formation of ν_r from μ_r is readily apparent.

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